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On the dynamic exponent of the two-dimensional Ising model

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Abstract. We use the method of 'damage spreading' to measure the time taken for the equilibrium damage (the magnetization) to disappear in finite systems ($10 \leqslant L \leqslant 103$) at the critical temperature of the infinite system. This time is found to scale as L^z with an exponent z of 2.16 ± 0.02. This value is compatible with the expected value of the dynamic exponent for the 2D Ising model, but we argue that this method is unable to measure fluctuations properly which we show are responsible for the higher values recently reported in the literature. We suggest that the slowest mode of relaxation has an exponent of $z \geqslant 2.27$.

1. Introduction

Our understanding of the static critical properties is on a fairly sound level, especially due to renormalization group (RG) theory both as a general framework for determining classes of systems with the same critical properties (those systems that flow to the same non-trivial fixed point) and as a calculational tool [1, 2]. Results from the RG transformation have successfully questioned accepted values from the highly accurate series expansion [3]. Our understanding and results for the dynamic exponents of critical systems are not as advanced [4]. Recent work [5–8], for example, suggests that the dynamic exponents z for the Potts models ($q = 2, 3, 4$) are at best only weakly dependent on q and in fact, the suggestion has been made that z might be independent of q [7, 8]! If this is true then it seems that the landscape in Hamiltonian space for fixed points may be decoupled in that the static properties may flow to different unstable static fixed points (the static universality classes of the Potts model are dependent on q) but the dynamic fixed points may, in fact, be the same and only dependent on the type of dynamics, e.g. Glauber, Kawasaki. This would be in direct conflict with the predictions of RG [4] which suggest that the fixed points are determined by the static and dynamic properties (in the form of conservation laws) of the system.

The static exponents for the 2D Ising model are known exactly but there is still debate on the value of the dynamic exponent z. High-powered Monte Carlo, Monte Carlo RG and other methods [7–24] suggest a value of ~ 2.13 whereas recent series analysis [25] and a 'damage spreading' [26] approach suggest a value of between 2.24 and 2.33. Some earlier work had also suggested larger values for the dynamic exponent. We have attempted to measure the dynamic exponent from the relaxation of the autocorrelation function of the order parameter. We find it difficult to extract a reliable region where this function shows pure exponential decay with time. We also considered the time taken for the magnetization, as measured by 'damage', to relax to 0 at T_c and the scaling behaviour of this quantity with system size gives an exponent of 2.16.

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Section 2 reviews the standard time evolution of the Ising model and we then introduce the concept of 'damage' and its definition for envolving to the equilibrium magnetization. We consider the time evolution of the autocorrelation function of the magnetization but find that it is difficult to extract the exponential region from the curve. We next consider the time taken for the 'damage' (magnetization) to go to 0 at T_c. We find a dynamic exponent of 2.16, a value compatible with the accepted value. We compare this method of measuring the dynamic exponent with the method of Poole and Jan [26] and show that the latter is able to take into account the rather strong fluctuations expected in two dimensions. These fluctuations are responsible for the larger dynamic exponent. We conclude that z is greater than 2.16 and most likely has a value of $\sim 2.27 \pm 0.05$.

2. Time-dependent phenomenon

The Hamiltonian for the Ising model contains no explicit or implicit time-dependent features. It is simply a potential function. We may formulate a time-dependent operator which has the desired property of sampling the equilibrium phase space of the model [12], which we perform below.

Our goal is to construct a time evolution operator Λ such that

$$\lim_{t\to\infty} \Lambda p[\sigma] = p_{eq}[\sigma]. \tag{1}$$

Consider the following master equation:

$$\frac{\partial(P[\sigma],t)}{\partial t} = \sum_{[\sigma]'} \{P([\sigma]',t)\,W([\sigma]'\to[\sigma]) - P([\sigma],t)\,W([\sigma]\to[\sigma]')\}. \tag{2}$$

This equation describes the rate of change of the probability of the state $[\sigma]$ as equal to the probability of the state $[\sigma']$ times the transition rate from the state $[\sigma']$ to the state $[\sigma]$, i.e. the probability of evolving from some other state to the state in question minus the probability of evolving out of the state in question (probability of the state $[\sigma]$ times the transition rate from the state $[\sigma]$ to $[\sigma']$). The equilibrium distribution is stationary in time

$$\frac{\partial P_{eq}[\sigma]}{\partial t} = 0 \tag{3}$$

and this implies that

$$P_{eq}[\sigma']\,W([\sigma']\to[\sigma]) = P_{eq}[\sigma']\,W([\sigma]\to[\sigma']) \tag{4}$$

$$\frac{P_{eq}[\sigma]}{P_{eq}[\sigma']} = \frac{W([\sigma']\to[\sigma])}{W([\sigma]\to[\sigma'])} = \frac{e^{-H[\sigma]/kT}}{e^{-H[\sigma']/kT}}. \tag{5}$$

If $[\sigma]\equiv[\sigma_i]$ and $[\sigma']\equiv[-\sigma_i]$ then

$$\frac{P_{eq}[\sigma_i]}{P_{eq}[-\sigma_i]} = \frac{e^{-E(\uparrow)/kT}}{e^{-E(\downarrow)/kT}} \tag{6}$$

where $E(\uparrow)$ is the energy of the system with the ith spin up and $E(\downarrow)$ is the energy of the system with the ith spin down. These conditions are satisfied if

$$W([-\sigma_i]\to[\sigma_i])=\frac{e^{-E(\uparrow)/kT}}{e^{-E(\uparrow)/kT}+e^{-E(\downarrow)/kT}} \tag{7}$$

which is the transition probability from $-\sigma_i$ to σ_i. We implement these transition probabilities in a particular way—whenever a site is visited the state of this site is ignored and the probability considered is that of finding the central site in the up state. If the random number is less than this probability then it is set up, otherwise, it is set down. This is referred to in the literature as heat-bath dynamics [27].

The evolution of the probability distribution is

$$\frac{\partial P([\sigma],t)}{\partial t}=\Lambda P([\sigma],t) \tag{8}$$

and its solution is

$$P([\sigma],t)=e^{\Lambda t}P([\sigma],0) \tag{9}$$

or

$$P([\sigma],t)=P_{eq}[\sigma]+a_1\,e^{\lambda_1 t}P_1[\sigma]+a_2\,e^{\lambda_2 t}P_2[\sigma]+\ldots \tag{10}$$

where $P_i[\sigma]$ are the eigenvectors and λ_i are the eigenvalues of Λ and a_i are the coefficients that define the initial probability distribution. The associated eigenvalue to P_{eq} is $\lambda_0=0$. For a finite system all other eigenvalues are finite and negative so that the system eventually evolves to equilibrium. The slowest mode of relaxation is $-\lambda_1=1/\tau_1$ and dynamic scaling asserts that $\tau_1\sim L^z$, where L is the linear size of the system and z is the dynamic critical exponent.

3. Damage

The concept of damage spreading has been used to test the stability of cellular automata to small perturbations [28, 29]. Two identical systems are constructed (same rules and same initial states). A small perturbation is introduced either by changing the rules or the states of a few sites [30] in one system and the time evolution of the Hamming distance is monitored. If the Hamming distance remains small and/or localized the system is considered to be stable; otherwise it is considered to be chaotic. This technique has also been applied to thermal systems [24, 27, 31, 32] where the same random numbers are now used in the update of the spins in the two systems.

Coniglio *et al* [33] drew attention to the fact that there were two distinct types of damage at a site i: d_i^{+-} and d_i^{-+}. d^{+-} represents the case where the spin at site i is up in system A and down in system B, while d^{-+} represents the opposite case, namely that the spin at site i is down in system A and up in system B. These authors also noted that for an appropriate initial damage there exist exact relationships between the equilibrium damage difference as defined above and thermodynamic quantities.

Consider the case where the boundary spins in system A are kept permanently in the up state while the equivalent boundary spins in system B are kept permanently down. We allow the systems to equilibrate under these conditions with heat-bath dynamics and consider the damage difference of the sites, excluding the boundary sites. Let $P_\uparrow^A$ be the probability that the origin is up in system A and $P_\uparrow^B$ the probability that the origin is up in system B. P is the probability that the origin is damaged, then $P = P^A_\uparrow - P^B_\uparrow$.

We define the following operators:

$$\pi_0 = \tfrac{1}{2}(1+\sigma_0) \qquad \bar{\pi}_0 = \tfrac{1}{2}(1-\sigma_0) \tag{11}$$

$$P^A_\uparrow = \frac{\langle \pi_0 \Pi_{i\in b}\pi_i\rangle}{\langle \Pi_{i\in b}\pi_i\rangle} \tag{12}$$

where $\langle\ldots\rangle$ is the thermal average of system A and b refers to the boundary spins and

$$P^B_\uparrow = \frac{\langle \pi_0 \Pi_{i\in b}\bar{\pi}_i\rangle}{\langle \Pi_{i\in b}\bar{\pi}_i\rangle} \tag{13}$$

where $\langle\ldots\rangle$ refers to the thermal average of system B.

From the symmetry of the Hamiltonian

$$P^A_\downarrow = \frac{\langle \bar{\pi}_0 \Pi_{i\in b}\pi_i\rangle}{\langle \Pi_{i\in b}\pi_i\rangle} = P^B_\uparrow = P^A_\uparrow. \tag{14}$$

Therefore $P = P^A_\uparrow - P^B_\uparrow - P^A_\uparrow - P^A_\downarrow = M$, where M is the magnetization of the system, since for a large system the effects of the boundary spins will be negligible on the central site. In other words, the damage difference leads to the magnetization.

4. The autocorrelation function and disappearance of the damage

The traditional means of measuring the characteristic time τ is by observing the autocorrelation function of the magnetization $C(t)$.

$$C(t) = \frac{\langle m(0)m(t)\rangle - \langle m\rangle^2}{\langle m^2\rangle - \langle m\rangle^2} = \sum_i A_i\, e^{-t/\tau_i}. \tag{15}$$

We measured the magnetization via the above-mentioned 'damage' method and a typical graph is shown in figure 1(a) for 100 000 independent trials and system size

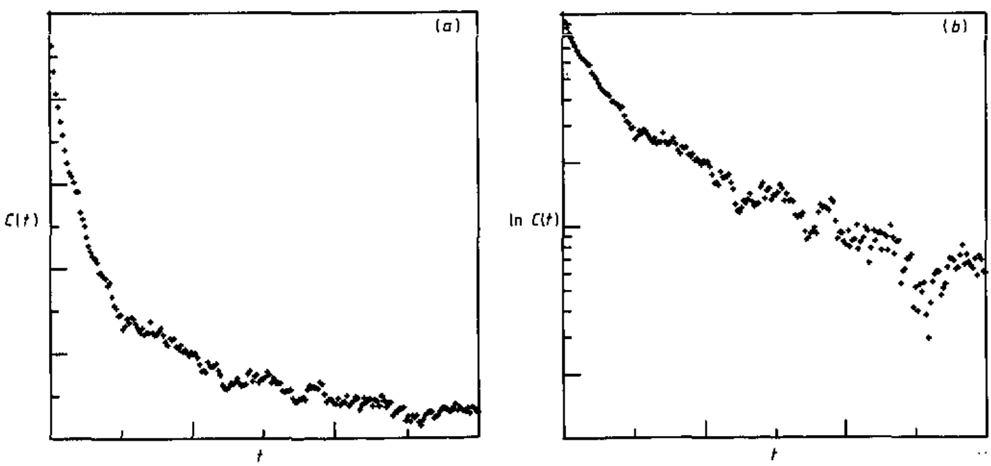

Figure 1. (a) Autocorrelation function of the magnetization $C(t)$ versus time for 100 000 independent trials and a system size of $L = 24$. (b) ln of the autocorrelation function $C(t)$ versus time.

$L = 24$. The ln of $C(t)$ versus t is shown in figure 1(b). We find it difficult to estimate $\tau(L)$ with a high degree of accuracy. Our results with this method would lead us to conclude that $z > 2$ but not much more.

Following a suggestion by Coniglio and Poole, we monitor the time taken for the equilibrium damage (i.e. the magnetization) to disappear once the boundary conditions are removed. We begin with this initial state (boundary sites permanently damaged) and allow it to evolve until the equilibrium damage has been reached. The damage which has occurred in the system due to the boundary sites leads as before, to the magnetization of the system.

The next step is to 'turn off' the influence from the boundary sites, that is, the boundary sites are no longer pinned and are replaced by periodic boundary conditions. We then allow both systems to evolve under the same heat-bath dynamics until eventually these systems have reached a point where there are absolutely no damaged sites anywhere in the system. In other words, the systems evolve until the magnetization (damage) relaxes from its equilibrium value to a value of zero. This time is directly related to the characteristic time. Figure 2 shows the ln–ln plot of the variation of this characteristic time with system size. We see a fairly good straight line from $L = 10$ to $L = 100$. Over 5000 trials were performed for the smaller systems and $\geqslant 1000$ for larger lattices. The slope of this graph, and hence z, is 2.16. We estimate an error margin of ± 0.02 from considering the maximum fluctuation of the results from various regions of L. We did not observe any systematic change in z with L. The value of $z = 2.16 \pm 0.02$ may be interpreted as follows: all values within the range 2.14–2.18 we consider to be

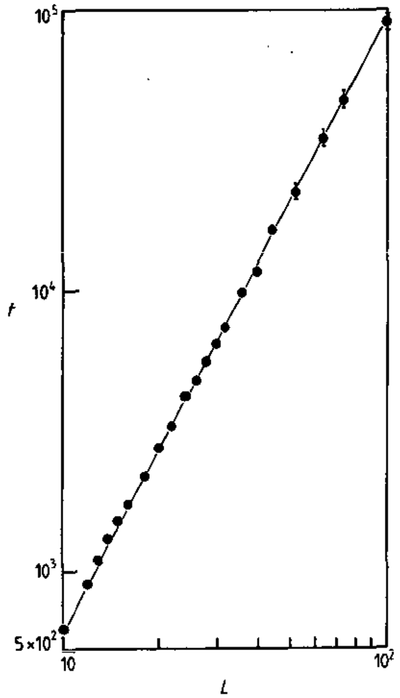

Figure 2. ln–ln plot of the time taken for the magnetization (damage) to disappear versus system size L ($10 \leqslant L \leqslant 103$). The error bars are shown by arrows when they are greater than the data points. The slope of the straight line is 2.16.

indistinguishable from our result, but we cannot completely rule out values of z in the ranges 2.12–2.14 or 2.18–2.20. However, our result is incompatible with values of z less than 2.12 or greater than 2.20.

5. Conclusion

The result $z = 2.16 \pm 0.02$ is in agreement with many of the recent results [6, 7] reported in the literature. Recently Poole and Jan [26] monitored the time taken for the damage to reach the edge of various size systems starting with a central site permanently damaged. They reported a value of 2.24 ± 0.04. Manna [34] repeated this work but with larger lattices, and found a value of z of 2.27 (no quoted error bars). We may ask: wherein lies the discrepancy between these higher values and the value of ~ 2.16?

Consider the spread of damage in two identical equilibrium systems apart from a permanently damaged central site. The damage will grow but due to large fluctuations it will often disappear, that is, it is reduced to a single damaged site before it eventually reaches the edge. This has been observed to occur several times before the damage eventually reaches the edge. In the above-mentioned work by Poole and Jan, the total time taken is recorded. Consider the case where we ignore the time taken in all previous attempts to spread the damage but only start from the last attempt from the single permanent damage and monitor this time for the successful reaching of the edge of the system. This is equivalent to the case considered here. An alternative way of appreciating this point is to consider the reverse case—when the damage has successfully reached the edge remove the constraint of a permanently damaged central site and measure the time taken for the damage to disappear. This is similar to the case we have considered for the disappearance of the magnetization. This would lead to a dynamic exponent of 2.16 because this approach is unable to measure the non-trivial fluctuations. We consider that these fluctuations are essential and would lead to an erroneous conclusion if ignored. We believe that the traditional methods are unable to account for these fluctuations. We draw the reader's attention to two further points:

(i) The recent work of Rogiers and Indekeu [25] from an extension of the series expansion report a value of 2.33 which we consider as an independent confirmation of our conclusion.

(ii) Poole and Jan [26] report a value of z for the 3D Ising model of 2.02 ± 0.01 which is in excellent agreement with the ε-expansion and other methods. This method is trivially exact for the 1D Ising model as the damaged site simply performs a random walk and hence z is 2.

Our main conclusion is that methods which cannot take into account the important fluctuations present in the 2D Ising model will find a value of z of ~ 2.16. We believe that unless the various different numerical methods agree then this problem is still open and may perhaps be resolved when more terms have been added to the series expansion. We consider 2.16 to be a lower bound for the dynamic exponent z and believe that the exact value is closer to 2.27.

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